Basic Probability Concepts and Discrete Probability Distributions

- Short Overview of Probability Concepts
  - Experiments, Events and Their Probability
  - Assigning Probabilities
  - Some Basic Relationships of Probability
  - Conditional Probability

  - Reading: Chapter 4 (we will pick up concepts from different sections)

- Discrete Probability Distributions

  - Reading: Chapter 5 (Sections 5.1-5.2)

Uncertainties

Managers often base their decisions on an analysis of uncertainties such as:

- What are the *chances* that sales will decrease if we increase prices?
- What is the *likelihood* a new assembly method will increase productivity?
- What are the *odds* that a new investment will be profitable?

Probability

Probability is a numerical measure of the likelihood that an event will occur

Probability values are always assigned on a scale from 0 to 1

A probability near zero indicates an event is quite unlikely to occur, near one - an event is almost certain to occur

Some Definitions

An experiment is any process that generates well-defined outcomes

Even though statistical experiment is repeated in exactly the same way, an entirely different outcome may occur

For this reason, statistical experiments are sometimes called *random experiments*

The sample space for an experiment is the set of **all** experimental outcomes

An ***event*** is a collection of outcomes

Some Definitions

Experiment	Experiment Outcomes
Toss a coin	Head, tail
Roll a die	1, 2, 3, 4, 5, 6
Conduct a sales call	Purchase, no purchase
Play a football game	Win, lose, tie
Hitting shuffle on iTunes library with 1000 songs	Any of 1000 songs that plays as a result of hitting shuffle

In each example above, the set of outcomes constitutes the *sample space* (because it lists all possible outcomes)

Assigning Probabilities (Basic Rules)

1. The probability assigned to each experimental outcome must be between 0 and 1 (inclusively):

   $$0 \leq P(A_i) \leq 1 \text{ for all } i$$

   where A_i is the ith experimental outcome and $P(A_i)$ is its probability

2. The sum of the probabilities for ALL experimental outcomes must equal 1:

   $$P(A_1) + P(A_2) + \ldots + P(A_n) = 1$$

   where n is the number of experimental outcomes

Assigning Probabilities

Classical Method: Assigning probabilities based on the assumption of equally likely outcomes

$$P(A) = \frac{\text{Number of possible outcomes that constitute Event } A}{\text{Total number of possible outcomes in the sample space}}$$

Empirical Method: Assigning probabilities based on experimentation or historical data

$$P(A) = \frac{\text{Frequency in which Event } A \text{ occurs}}{\text{Total number of observations}}$$

Subjective Method: Assigning probabilities based on judgment or experience

Law of Large Numbers (LLN)

Important for finding probabilities and allows us predict how events will play out in the long-run:

As the number of trails or observations increases, the observed probability of an event (empirical probability) approaches theoretical (classical) probability

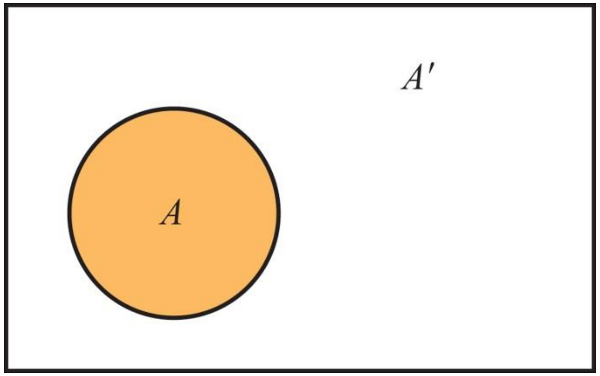
For example:

- Flip of a fair coin → P(Heads) = 0.5 is a theoretical probability
- However, if we flip a coin 10 times, we do not necessarily get exactly 5 heads (proportion of Heads ≠ 0.5)
- According to the LLN, as we continue flipping a coin, the proportion of Heads should get closer to 0.5

Complement of an Event

The **complement** of event A is defined to be the event consisting of all outcomes that are not in A

The complement of A is denoted by A'



Sample Space S

Venn Diagram

$$P(A) + P(A') = 1$$

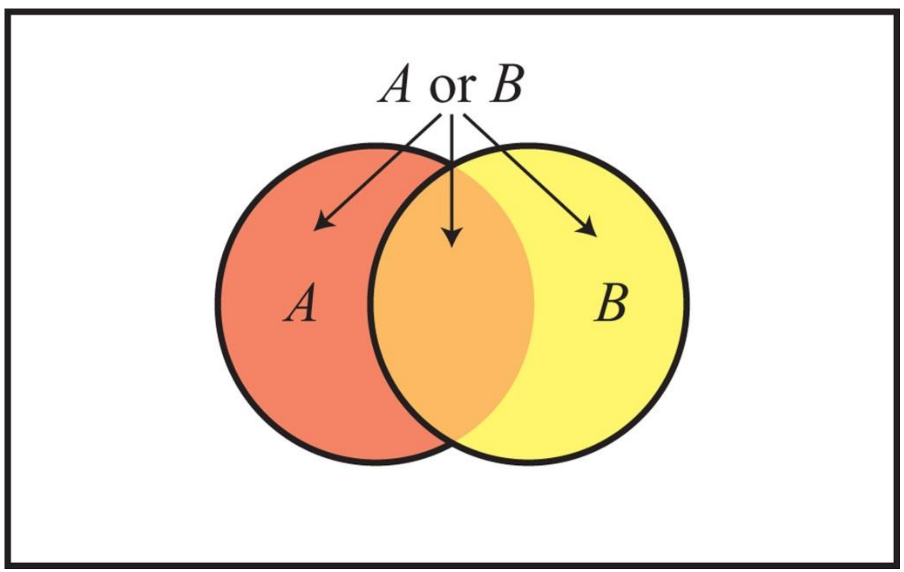
or

$$P(A) = 1 - P(A')$$

The Union of Events

The union of events A and B is the event containing all outcomes that are in A or B or both

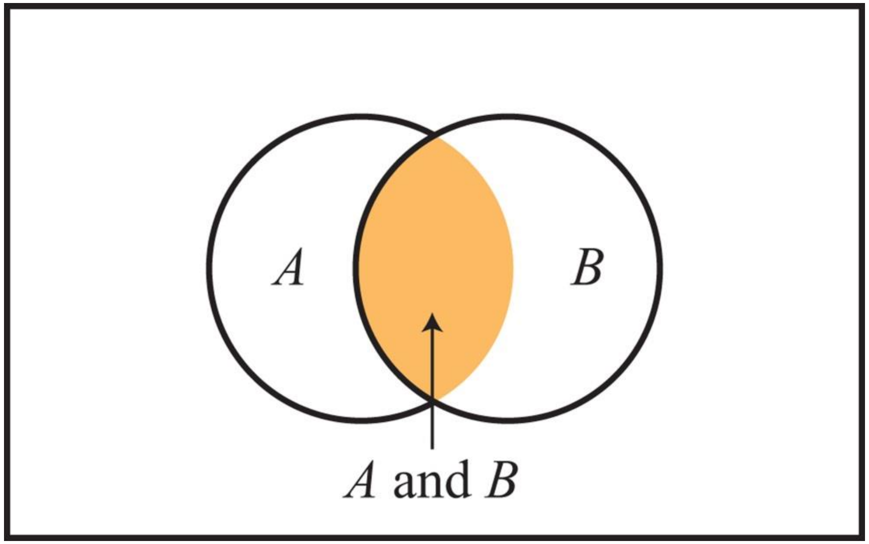
The union of events A and B is denoted by $A \cup B$



The Intersection of Events

The intersection of events A and B is the set of all outcomes that are in both A and B

The intersection of events A and B is denoted by $A \cap B$

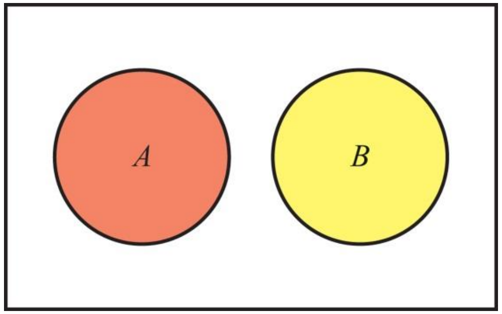


Mutually Exclusive Events

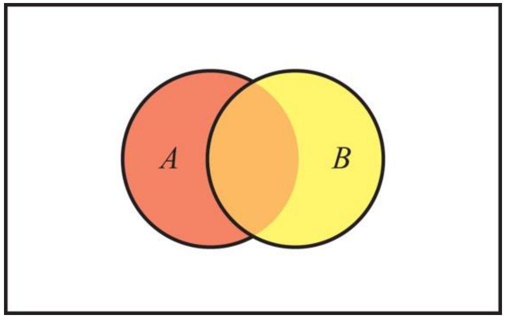
Two events are said to be mutually exclusive if the events have no outcomes in common

Two events are mutually exclusive if, when one event occurs, the other cannot occur

mutually exclusive



not mutually exclusive



The Addition Rule

The addition rule provides a way to compute the probability of event A, or B, or both A and B occurring:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

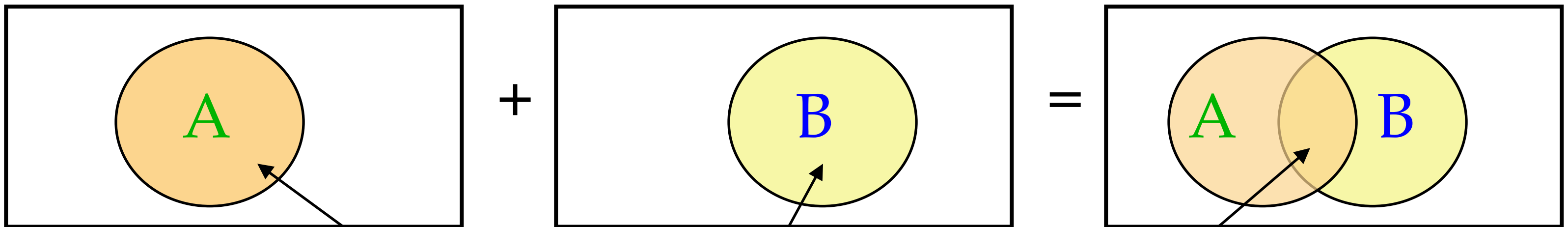
If events A and B are mutually exclusive, $P(A \cap B) = 0$

$$P(A \text{ or } B) = P(A) + P(B)$$

or

$$P(A \cup B) = P(A) + P(B)$$

The Addition Rule



$$P(A \text{ or } B) = P(A) + P(B) - \underbrace{P(A \text{ and } B)}_{\text{Don't count common elements twice}}$$

Note: $P(A \text{ and } B) = 0$ if events A and B are mutually exclusive

Conditional Probability

The probability of an event given or knowing that another event has occurred is called a <u>conditional probability</u>

The conditional probability of <u>A given B</u> is denoted by $P(A \mid B)$ and computed as:

$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

or

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Independent and Dependent Events

Two events are considered **independent** if the occurrence of one event has no impact on the occurrence of the other event

If Events A and B are independent:

$$\boldsymbol{P(A \mid B) = P(A)}$$

If $P(A \mid B) \neq P(A)$, then events A and B are not independent

The Multiplication Rule

The **multiplication rule** is used to determine the probability of the intersection (joint probability) of two events occurring, or $P(A \text{ and } B)$

Multiplication rule **for dependent events**:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

or

$$P(A \cap B) = P(B)P(A|B)$$

The Multiplication Rule

Multiplication rule **for two independent events**:

$$P(A \cap B) = P(B)P(A)$$

- Since events A and B are independent $\Rightarrow P(A \mid B) = P(A)$

When **multiple events** are all independent, the probability of them all occurring is simply the product of their individual probabilities:

$$P(A_1 \text{ and } A_2 \text{ and} \ldots \text{and } A_n) = P(A_1)P(A_2) \ldots P(A_n)$$

Mutual Exclusiveness and Independence

Mutually exclusive events ≠ independent events

- If one mutually exclusive event is known to occur, the other cannot occur
  - ⇒ thus, the probability of the other event occurring is reduced to zero (and they are therefore dependent)

Discrete Probability Distributions

- Random Variables
- Discrete Probability Distributions
- Expected Value and Variance
- Binomial Probability Distribution

- Reading: Chapter 5 (Section 5.1)

Random Variables

A <u>random variable</u> is a *numerical* description of the outcome of an experiment

A <u>discrete random variable</u> may assume either a finite number of values or an infinite sequence of values

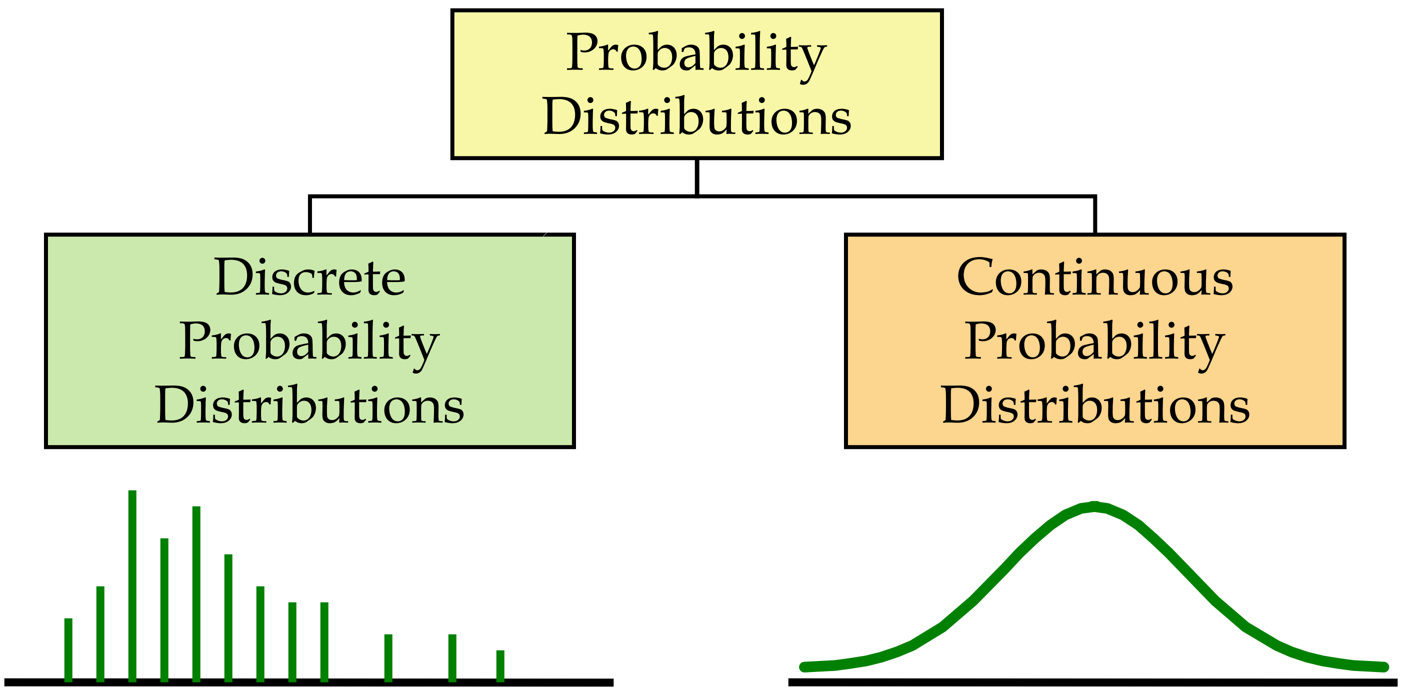
- Values are whole numbers (integers), usually counted
  - number of complaints per day; number of TVs in a household

A <u>continuous random variable</u> may assume any numerical value in an interval or collection of intervals

- Often measured, fractional values are possible
  - time required to complete a task; height; distance

Probability Distributions

Every random variable is associated with a probability distribution that describes the variable completely



Discrete Probability Distributions

A **discrete probability distribution** is

- a listing of all the possible outcomes of an experiment for a discrete random variable
- along with the relative frequency of each outcome

Discrete probability distribution:

- describes how probabilities are distributed over the values of the random variable
- can be represented by a table, a graph, or a formula
- a formula used to describe discrete probability distribution is called a **probability function**, denoted by $P(x)$ (sometimes $f(x)$), which provides the probability for each value of the random variable x

Discrete Probability Distribution

Experiment: Rolling a six-sided die

Let x = the number showing on the die

The probability distribution for the random variable x is:

x	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

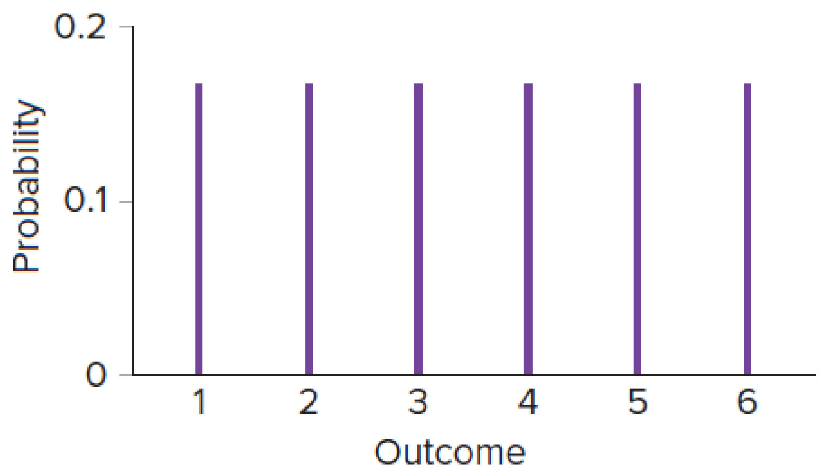
- This table:
  1. Lists all possible outcomes of the experiment
  2. Each outcome has an associated probability of 1/6

⇒ The pairs of values and their probabilities form the **probability distribution** for the random variable x

Discrete Probability Distribution

A probability distribution may be shown graphically:

- The values of x are placed on the horizontal axis and the probabilities $P(x)$ on the vertical axis
- A bar is drawn so that its height equals $P(x)$
- E.g, this graph represents the six-sided die experiment:



Rules for Discrete Probability Distributions

- The probability of each value of x, $P(x)$, must be between 0 and 1 (inclusive):

$$0 \leq P(x) \leq 1 \quad \text{for all values of } x$$

- The sum of the probabilities for all value of x in the distribution must be 1:

$$\sum_{i=1}^{n} P(x_i) = 1$$

where n equals the total number of possible values

Mean of a Discrete Probability Distribution

The **mean**, $\boldsymbol{\mu}$, of a discrete probability distribution is the weighted average of ALL values of the random variable

- The weights are the probabilities
- The mean does not have to be a value the random variable can assume

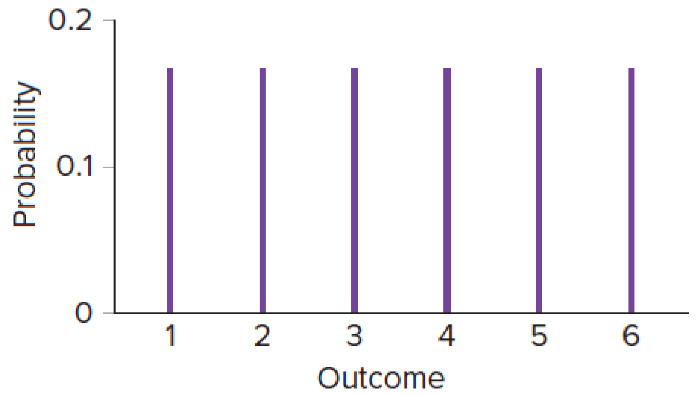
Mean is also known as the **expected value** $\boldsymbol{E(x)}$

$$E(x) = \mu = \sum_{i=1}^{n} x_i P(x_i)$$

where: μ = The mean of the discrete probability distribution
x_i = The value of the random variable for the i^{th} outcome
$P(x_i)$ = The probability that the i^{th} outcome will occur
n = The number of outcomes in the distribution

Mean of a Discrete Probability Distribution

Calculating the mean ($\mu = \sum_{i=1}^{n} x_i P(x_i)$):



Probability Distribution

x_i	$P(x_i)$	$x_i P(x_i)$
1	1/6	1/6
2	1/6	2/6
3	1/6	3/6
4	1/6	4/6
5	1/6	5/6
6	1/6	6/6
Total	**1**	**3.5** $= \sum_{i=1}^{n} x_i P(x_i)$

The expected value of a die roll is 3.5

Variance and SD of a Discrete Probability Distribution

Variance summarizes the variability in the values of a random variable

$$\sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 P(x_i)$$

where: σ^2 = The variance of the discrete probability distribution
x_i = The value of the random variable for the i^{th} outcome
μ = The mean of the discrete probability distribution
$P(x_i)$ = The probability that the i^{th} outcome will occur
n = The number of outcomes in the distribution

The **standard deviation** is the square root of the variance

$$\sigma = \sqrt{\sigma^2}$$

Variance and SD of a Discrete Probability Distribution

Calculating the variance and SD:

x_i	$P(x_i)$	μ	$(x_i - \mu)$	$(x_i - \mu)^2$	$(x_i - \mu)^2 P(x_i)$
1	1/6	3.5	-2.5	6.25	1.0417
2	1/6	3.5	-1.5	2.25	0.3750
3	1/6	3.5	-0.5	0.25	0.0417
4	1/6	3.5	0.5	0.25	0.0417
5	1/6	3.5	1.5	2.25	0.3750
6	1/6	3.5	2.5	6.25	1.0417
Total	**1**				**2.9167**

Variance

The standard deviation is $\sigma = \sqrt{2.9167} \approx 1.7$

Excel Exercise >>

Discrete Probability Distributions

- Reading: Chapter 5 (Section 5.2)

Specific Discrete Probability Distributions

- Binomial
- Poisson → Won’t talk about it in class
- Hypergeometric → Won’t talk about it in class

Binomial Distribution

A **binomial random variable** is defined as the number of successes denoted by x achieved in the n trials and is a result of a Binomial Experiment

Characteristics of a Binomial Experiment:

1. The experiment consists of a fixed number of (Bernoulli) trials, denoted by n
2. Each trial has only two possible outcomes, a success and a failure
3. The probability of a success p and the probability of a failure q are constant throughout the experiment
4. Each trial is independent of the other trials in the experiment

Binomial Distribution

Examples of binomial settings:

- Flip of a coin: H or T
- Perspective student either accepts or rejects university admission
- A bank grants or denies a loan to a mortgage application
- A customer uses or does not use a credit card when making a purchase
- A college graduate applies or does not apply to graduate school

Binomial Distribution

The binomial probability function calculates the probability of a specific number of successes (x) for a certain number of trials (n), given specified probability of success (p) and probability of failure (q)

$$P(x, n) = \frac{n!}{(n-x)!\,x!} p^x q^{n-x}$$

where: $P(x, n)$ = The probability of observing x successes in n trials
n = Number of trials
x = Number of successes
p = Probability of a success
q = Probability of a failure

Mean and SD of a Binomial Distribution

Mean of a Binomial Distribution

$$\mu = np$$

Variance of a Binomial Distribution

$$\sigma^2 = npq$$

Standard Deviation of a Binomial Distribution

$$\sigma = \sqrt{npq}$$

where: n = The number of trials
p = The probability of a success
q = The probability of a failure

Excel Exercise >>

Excel Time: Exercise (Not in the Textbook)

The number of homes that a realtor sells over a one-month period has the probability distribution shown below:

Number of Houses Sold	*Probability*
0	0.06
1	0.11
2	0.24
3	0.27
4	0.2
5	0.12

1. Verify that this is a legitimate probability distribution.
2. Find the mean of the distribution.
3. Find the variance and the SD of the distribution.
4. Find the probability that the realtor sells 2 houses or less.
5. Find the probability that the realtor sells less than 2 houses.
6. Find the probability that the realtor sells 3 or 4 houses.

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Excel Time: Exercise 5.18

According to a survey by Statista from February 2016, about 35% of Snapchat users are 18 to 24 years old. Consider a random sample of nine Snapchat users .

a. What is the probability that **exactly three** users from this sample are 18 to 24 years old?

b. What is the probability that **less than four** users from this sample are 18 to 24 years old?
   - Use BINOM.DIST.RANGE() or BINOM.DIST() function

c. What is the probability that **six or seven** users from this sample are 18 to 24 years old?
   - Use BINOM.DIST.RANGE() or BINOM.DIST()

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Excel Time: Exercise 5.18, Continued I

Repeat parts b and c of Exercise 5.18. But calculate corresponding probabilities in a different way:

- First, **generate the entire binomial distribution** using BINOM.DIST() and verify that it is a legitimate probability distribution.
- Build a histogram for this distribution. Make sure that you can interpret it.
- Now, find the probabilities from parts b and c WITHOUT calling BINOM.DIST() function.
- Make sure that your computations provide the same answers as before.

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Excel Time: Exercise 5.18, Continued II

Compute the mean and the SD of the distribution:

- First, use the shortcut formulas for the binomial distribution.
- Now, use the binomial distribution that you built earlier and general formulas for the mean and the variance of the discrete probability distribution.
- Make sure that all computations provide the same answers.

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Excel Time: Binomial Probabilities

Excel's **BINOM.DIST()** function can be used to find **binomial probabilities**

The BINOM.DIST() function has the following format:

=BINOM.DIST(x, n, p, cumulative)

where: x = Number of successes
n = Number of trials
p = Probability of a success
cumulative = FALSE or 0, if you want to determine the probability of **exactly** $\boldsymbol{x}$ successes occurring
cumulative = TRUE or 1, if you want to determine the probability of $\boldsymbol{x}$ **or fewer** successes occurring

Excel Time: Binomial Probabilities

BINOM.DIST.RANGE() function calculates the probability for the number of successes falling into a specified range.

The function has the following format:

$$= \text{BINOM.DIST.RANGE}(\, n, p, a, b \,)$$

where:

n = Number of trials
p = Probability of a success
a = The minimum number of successes that you want to calculate the probability for (must be $\leq n$)
b = The maximum number of successes that you want to calculate the probability for:

- If provided, b must be $a \leq b \leq n$.
- If b argument is omitted, the function calculates the probability of exactly a successes.

Excel Time: Example of Using Excel Functions

Suppose, we toss a coin 100 times (n = 100). In a single flip, heads occur with probability 0.5 (p = 0.5). Find the following probabilities:

In words	Mathematical Expression	Excel Function
Probability of exactly 10 heads	$P(\boldsymbol{x = 10}, n = 100)$	=BINOM.DIST(10, 100, 0.5, **FALSE**)
Probability of 8 heads or less	$P(\boldsymbol{x \leq 8}, n = 100)$	=BINOM.DIST(8, 100, 0.5, **TRUE**) or =BINOM.DIST.RANGE(100, 0.5, **0**, **8**)
Probability of less than 5 heads	$P(x < 5, n = 100) = P(\boldsymbol{x \leq 4}, n = 100)$	=BINOM.DIST(**4**, 100, 0.5, **TRUE**) or =BINOM.DIST.RANGE(100, 0.5, **0**, **4**)
Probability of more than 5 heads	$P(x > 5, n = 100) = P(\boldsymbol{x \geq 6}, n = 100)$	=BINOM.DIST.RANGE(100, 0.5, **6**, **100**) or =1 - BINOM.DIST(**5**, 100, 0.5, 1)
Probability of 8 heads or more	$P(\boldsymbol{x \geq 8}, n = 100)$	=BINOM.DIST.RANGE(100, 0.5, **8**, **100**) or =1 - BINOM.DIST(**7**, 100, 0.5, 1)

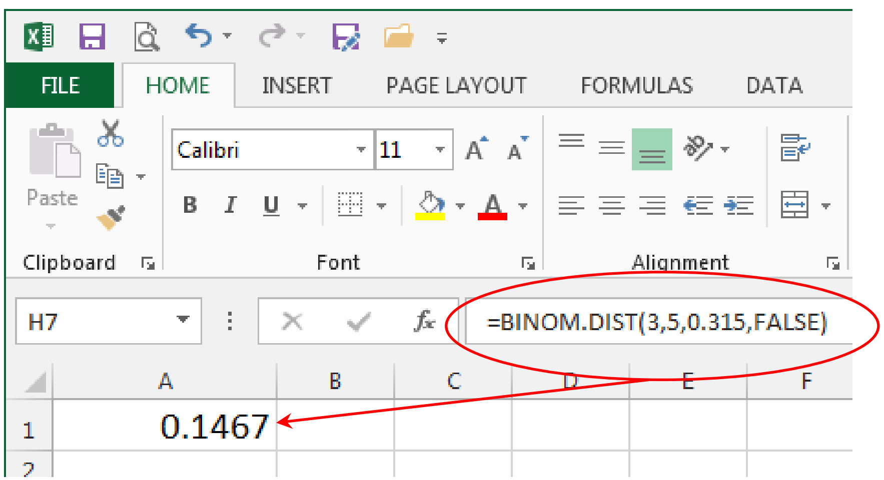
Excel Time: Calculate Binomial Probabilities

Example: In 2010, women made up 31.5% of all lawyers, according to the American Bar Association.

1. What is the probability that in a randomly selected group of 5 lawyers, exactly 3 are women?

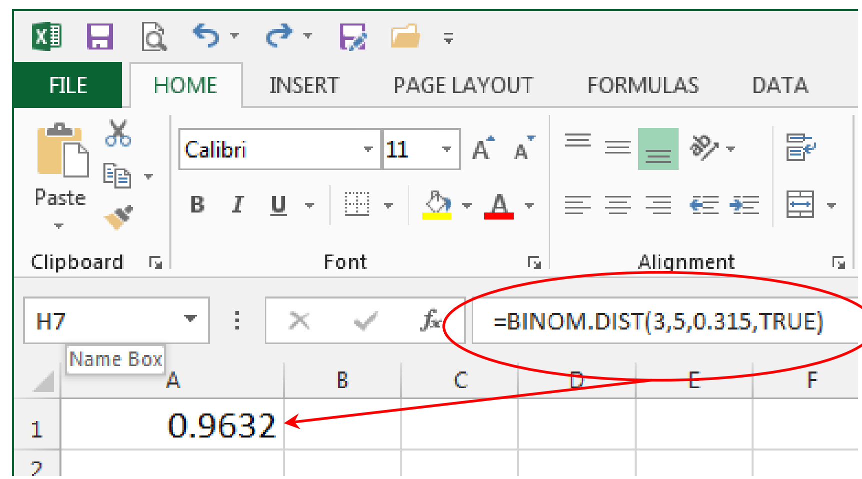
2. What is the probability that 3 or fewer are women?

1.



$P(x = 3) = 0.1467$

2.



$P(x \leq 3) = 0.9632$

Excel Time: Exercise 5.20 (Extra Practice)

An e-commerce Web site claims that 6% of people who visit the site make a purchase. Answer the following questions based on a random sample of 15 people who visited Web site.

a. What is the probability that none of the people will make a purchase?
b. What is the probability that LESS than three people will make a purchase?
c. What is the probability that MORE than one person will make a purchase?
d. What is the means and SD of this distribution?

Excel Time: Exercise 5.15 (Extra Practice)

A recent survey taken by Adecco of 1,047 employees found that 28% of them would lay off their bosses if they could. Consider a random sample of 10 employees.

a. What is the probability that **exactly five** employees would lay off their bosses?
b. What is the probability that **three or fewer** employees would lay off their bosses?
c. What is the probability that **five or more** employees would lay off their bosses?
d. What is the means and SD of this distribution?

Excel Time: Exercise 5.14 (Extra Practice)

A student estimated that the probability of correctly answering each question in a multiple-choice question is 80%. Using Excel, determine the probability of earning at least a 70% grade on a 20-question exam.

Excel Time: Exercise 5.52 (Extra Practice)

In 2017, 80% of Southwest Airlines arrived at their destinations on time. Suppose a random sample of 12 Southwest Airlines flights was selected.

a. What is the probability that all 12 flights were on time?
b. What is the probability that MORE than nine flights were on time?
c. What is the probability that seven or fewer flights were on time?
d. What is the means and SD of this distribution?
e. Construct a histogram for this distribution.